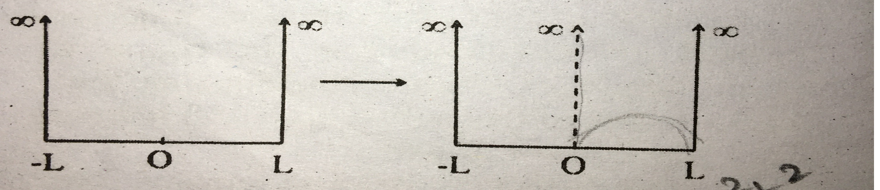
Q:

A quantum particle of mass m in one dimension, confined to a rigid box as shown in the figure, is in its ground state. An infinitesimally thin wall is very slowly raised to infinity at the centre of the box, in such a way that the system remains in its ground state at all times. Assuming that no energy is lost in raising the wall, the work done on the system when the wall is fully raised, eventually separating the original box into two compartments, is



1. $\frac{3\pi^2\hbar^2}{8mL^2}$

2. $\frac{\pi^2\hbar^2}{8mL^2}$

3. $\frac{\pi^2\hbar^2}{2mL^2}$

4. 0

Answer: Option a(Source Y. K. Lim)

An infinitely deep one-dimensional square well potential confines a particle to the region $0 \leq x \leq L$. Sketch the wave function for its lowest energy eigenstate. If a repulsive delta function potential, $H' = \lambda\delta(x - L/2)$ $(\lambda > 0)$, is added at the center of the well, sketch the new wave function and state whether the energy increases or decreases. If it was originally E_0, what does it became when $\lambda \to \infty$?

(Wisconsin)

Solution:

For the square well potential the eigenfunction corresponding to the lowest energy state and its energy value are respectively

$$\phi_0(x) = \sqrt{2/L}\sin(\pi x/L),$$

$$E_0 = \frac{\pi^2\hbar^2}{2mL^2}.$$

A sketch of this wave function is shown in Fig. 1.12

With the addition of the delta potential $H' = \lambda\delta(x - L/2)$, the Schrödinger equation becomes

$$\psi'' + [k^2 - \alpha\delta(x - L/2)]\ \psi = 0,$$

where $k^2 = 2mE/\hbar^2, \alpha = 2m\lambda/\hbar^2$. The boundary conditions are

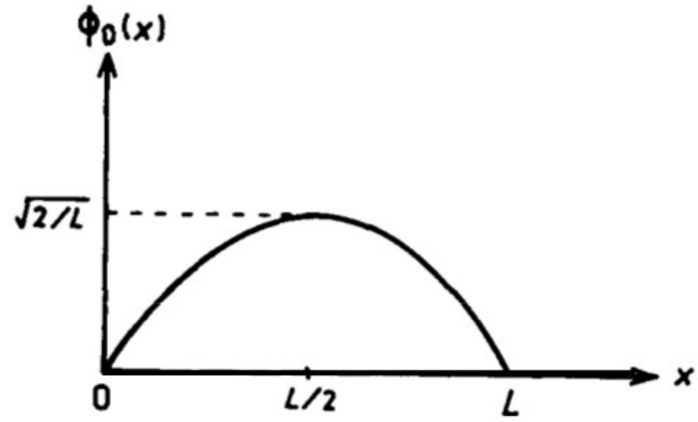


Fig. 1.12

$$\psi(0) = \psi(L) = 0\,, \tag{1}$$

$$\psi'\left[\left(\frac{L}{2}\right)^+\right] - \psi'\left[\left(\frac{L}{2}\right)^-\right] = \alpha\psi(L/2)\,, \tag{2}$$

$$\psi\left[\left(\frac{L}{2}\right)^+\right] = \psi\left[\left(\frac{L}{2}\right)^-\right]. \tag{3}$$

Note that (2) arises from taking $\lim_{\varepsilon\to 0}\int_{\frac{L}{2}-\varepsilon}^{\frac{L}{2}+\varepsilon} dx$ over both sides of the Schrödinger equation and (3) arises from the continuity of $\psi(x)$ at $x = \frac{L}{2}$.

The solutions for $x \neq \frac{L}{2}$ satisfying (1) are

$$\psi = \begin{cases} A_1 \sin(kx), & 0 \le x \le L/2, \\ A_2 \sin[k(x - L)], & L/2 \le x \le L. \end{cases}$$

Let $k = k_0$ for the ground state. Condition (3) requires that $A_1 = -A_2 = A$, say, and the wave function for the ground state becomes

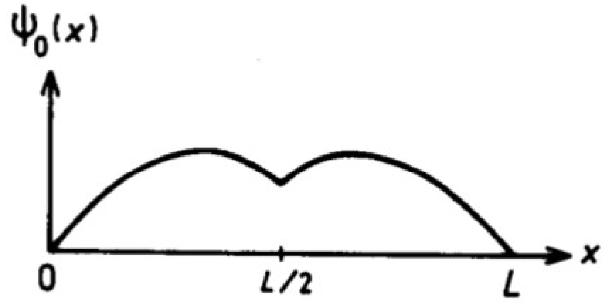
$$\psi_0(x) = \begin{cases} A \sin(k_0 x), & 0 \le x \le L/2, \\ -A \sin[k_0(x - L)], & L/2 \le x \le L. \end{cases}$$

Condition (2) then shows that k_0 is the smallest root of the transcendental equation

$$\cot(kL/2) = -\frac{m\lambda}{k\hbar^2} .$$

As $\cot\left(\frac{kL}{2}\right)$ is negative, $\pi/2 \leq k_0$ L/2 $\leq \pi$, or $\pi/L \leq k_0 \leq 2\pi/L$. The new ground-state wave function is shown Fig. 1.13. The corresponding energy is $E = \hbar^2 k_0^2/2m \geq E_0 = \frac{\pi^2\hbar^2}{2mL^2}$, since $k_0 \geq \frac{\pi}{2}$. Thus the energy of the new ground state increases.

Furthermore, if $\lambda \to +\infty, k_0 \to 2\pi/L$ and the new ground-state energy $E \to 4E_0$.



and the above equation becomes

$$\ddot{\rho} + \frac{3m_2 g}{(m_1 + m_2)(l - d)}\rho = 0 .$$

Hence ρ oscillates about O, i.e. r oscillates about the value $l - d$, with angular frequency

$$\omega = \sqrt{\frac{3m_2 g}{(m_1 + m_2)(l - d)}} ,$$

or period

$$T = 2\pi\sqrt{\frac{(m_1 + m_2)(l - d)}{3m_2 g}} .$$